Repeat with the remaining area terms. Finally, $\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\sin\frac{\alpha_3}{2}=\frac{[A_1A_2A_3]}{4Rs}$ is a known area formula, and so we have

$$\begin{aligned}\frac{[D_1U_1V_1]}{\sin^2\frac{\alpha_2-\alpha_3}{2}}+\frac{[D_2U_2V_2]}{\sin^2\frac{\alpha_3-\alpha_1}{2}}+\frac{[D_3U_3V_3]}{\sin^2\frac{\alpha_1-\alpha_2}{2}} &= \frac{[A_1A_2A_3]}{4Rs}\cdot 4R^2(\sin\alpha_1+\sin\alpha_2+\sin\alpha_3)\\ &= \frac{[A_1A_2A_3]}{4Rs}\cdot 4R^2(\tfrac{a_1}{2R}+\tfrac{a_2}{2R}+\tfrac{a_3}{2R}\\ &= \frac{[A_1A_2A_3]}{4Rs}\cdot 4R^2\cdot\frac{s}{R}\\ &= [A_1A_2A_3],\end{aligned}$$

concluding the proof.

Editor's Comments. It is interesting to observe that D_i, U_i, D_{i+1} and D_i, D_{i+1}, V_{i+1} are collinear (as can be shown either by analytic geometry, or by angle chasing). This was a key part of the proposer's solution, but not used in any of the other received solutions.

3948. *Proposed by George Apostolopoulos.*

Let $a_1, a_2, \dots a_n$ be real numbers such that $a_1 > a_2 > \dots > a_n$. Prove that

$$\frac{1}{a_1-a_2}+\frac{1}{a_2-a_3}+\cdots+\frac{1}{a_{n-1}-a_n}+a_1-a_n\geq 2(n-1).$$

When does the equality hold?

We received 25 submissions. We present the solution by Prithwijit De.

Let $b_k = a_k - a_{k+1}$ for $k = 1, 2, \dots, n-1$. Then the left-hand side of the inequality reduces to

$$\sum_{k=1}^{n-1}\left(\frac{1}{b_k}+b_k\right).$$

Notice that for all k, we have $b_k > 0$ and $\dfrac{1}{b_k}+b_k\geq 2$. Hence

$$\sum_{k=1}^{n-1}\left(\frac{1}{b_k}+b_k\right)\geq 2(n-1).$$

Equality occurs if and only if $b_k = 1$ for all k. That is if and only if $a_k = a_{k+1}+1$ for $k = 1, 2, \dots, n-1$.

3949. *Proposed by Arkady Alt.*

For any positive real a and b, find

$$\lim_{n\to\infty}\left((n+1)\left(\frac{a^{\frac{1}{n+1}}+b^{\frac{1}{n+1}}}{2}\right)^{n+1}-n\left(\frac{a^{\frac{1}{n}}+b^{\frac{1}{n}}}{2}\right)^{n}\right).$$

We received nine submissions, all of which were correct. We present the solution by Michel Bataille.

We prove that the limit is $\sqrt{ab}$.

Let $U_n = \left(\frac{a^{\frac{1}{n}}+b^{\frac{1}{n}}}{2}\right)^n$ and $V_n = (n+1)U_{n+1} - nU_n$.

First, we consider the case $a = b$. Then, $\frac{a^{\frac{1}{n}}+b^{\frac{1}{n}}}{2} = a^{\frac{1}{n}}$ for every positive integer n, so that

$$V_n = (n+1)a - na = a$$

and $\lim\limits_{n\to\infty} V_n = \sqrt{a^2}$ is obvious.

From now on, we suppose $a \neq b$. We adopt the notation α for $\ln\left(\frac{a}{b}\right)$.

As $n \to \infty$, we may write

$$\begin{aligned}
U_n &= \left(\frac{b^{\frac{1}{n}}\left(1+\left(\frac{a}{b}\right)^{\frac{1}{n}}\right)}{2}\right)^n \\
&= \frac{b}{2^n}\left(1+e^{\frac{\alpha}{n}}\right)^n \\
&= \frac{b}{2^n}\left(2+\frac{\alpha}{n}+\frac{\alpha^2}{2n^2}+o(1/n^2)\right)^n \\
&= b\left(1+\frac{\alpha}{2n}+\frac{\alpha^2}{4n^2}+o(1/n^2)\right)^n.
\end{aligned}$$

In consequence,

$$\begin{aligned}
\ln(U_n) &= \ln(b) + n\ln\left(1+\frac{\alpha}{2n}+\frac{\alpha^2}{4n^2}+o(1/n^2)\right) \\
&= \ln(b) + n\left(\frac{\alpha}{2n}+\frac{\alpha^2}{4n^2}-\frac{1}{2}\cdot\frac{\alpha^2}{4n^2}+o(1/n^2)\right) \\
&= \ln(b) + \frac{\alpha}{2} + \frac{\alpha^2}{8n} + o(1/n),
\end{aligned}$$

and so

$$U_n = be^{\alpha/2}e^{\alpha^2/8n+o(1/n)} = be^{\alpha/2}\left(1+\frac{\alpha^2}{8n}+o(1/n)\right).$$

As a result, we obtain

$$nU_n = nbe^{\alpha/2} + \frac{\alpha^2 be^{\alpha/2}}{8} + o(1)$$

and

$$V_n = (n+1)be^{\alpha/2} - nbe^{\alpha/2} + o(1) = be^{\alpha/2} + o(1).$$

We conclude that

$$\lim_{n\to\infty} V_n = be^{\alpha/2} = b\cdot\sqrt{\frac{a}{b}} = \sqrt{ab}.$$